A FAMILY OF NONLINEAR SCHRÖDINGER EQUATIONS: LINEARIZING TRANSFORMATIONS AND RESULTING STRUCTURE

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Abstract

We examine a recently-proposed family of nonlinear Schrödinger equations[4] with respect to a group of transformations that linearize a subfamily of them. We investigate the structure of the whole family with respect to the linearizing transformations, and propose a new, invariant parameterization.

1. INTRODUCTION

Previous work[1–5] on the representation theory of an infinite-dimensional kinematical algebra on $I\!R^3$, and the corresponding infinite-dimensional group, led to a Fokker-Planck type of equation for the quantum-mechanical probability density and current,

$$\partial_t \rho = -\vec{\nabla} \cdot \vec{j} + D \Delta \rho \,, \tag{1.1}$$

and in turn to a family $\mathcal{F}_D$ of nonlinear Schrödinger equations. $\mathcal{F}_D$ is parameterized by the classification parameter D of the unitarily inequivalent group representations (the diffusion coefficient in Eq. (1.1)), and five real model parameters $D'c_1, \ldots, D'c_5$:

$$i\hbar\partial_t\psi = \left(-\frac{\hbar^2}{2m}\Delta + V(\vec{x})\right)\psi + i\frac{\hbar D}{2}\frac{\Delta\rho}{\rho}\psi + \hbar D'\left(\sum_{j=1}^{5} c_j R_j[\psi]\right)\psi \,. \tag{1.2}$$

Here D' also has the dimensions of a diffusion coefficient (so that the c_j are dimensionless), and the nonlinear functionals R_j are complex homogeneous of degree zero, defined by:

$$\begin{gathered} R_1[\psi] := \frac{\vec{\nabla}\cdot\vec{J}}{\rho}\,, \qquad R_2[\psi] := \frac{\Delta\rho}{\rho}\,, \qquad R_3[\psi] := \frac{\vec{J}^2}{\rho^2}\,, \\ R_4[\psi] := \frac{\vec{J}\cdot\vec{\nabla}\rho}{\rho^2}\,, \qquad R_5[\psi] := \frac{(\vec{\nabla}\rho)^2}{\rho^2}\,, \end{gathered} \tag{1.3}$$

where $\rho := \bar{\psi}\psi$ and $\vec{J} := \mathrm{Im}\,(\bar{\psi}\vec{\nabla}\psi) = (m/\hbar)\vec{j}$.

A subfamily of these equations, characterized by $D'c_1 = D = -D'c_4$, together with $c_2 + 2c_5 = 0$ and $c_3 = 0$, satisfies Ehrenfest's theorem in quantum mechanics[4], and is linearizable via a nonlinear transformation[6, 7]. In this short note we sketch some ideas connected with the linearizing transformations; for a detailed derivation and description with emphasis on the physical interpretation, we refer to forthcoming articles[8].

2. LINEARIZATION

The members of the Ehrenfest subfamily $\mathcal{F}_D^{Ehr}$ can be transformed into linear Schrödinger equations by a transformation $\psi \mapsto \psi' = N(\psi)$, if the remaining unspecified model parameter $D'c_2$ satisfies

$$\frac{4m}{\hbar} D'c_2 \, < \, 1 - \frac{4m^2D^2}{\hbar^2} \, . \tag{2.1}$$

Here N depends on two real parameters $\gamma, \Lambda \in I\!R, \Lambda \neq 0$, and is given by

$$\psi' := N_{(\Lambda,\gamma)}(\psi) = \psi^{\frac{1}{2}(1+\Lambda+i\gamma)} \bar{\psi}^{\frac{1}{2}(1-\Lambda+i\gamma)} = |\psi| \, e^{i(\gamma \ln |\psi| + \Lambda \arg \psi)} \, . \tag{2.2}$$

This maps solutions of the Ehrenfest subfamily of (1.2) into solutions of the linear Schrödinger equation,

$$i\frac{\hbar}{\Lambda}\partial_t \psi' = \left(-\frac{\hbar^2}{2\Lambda^2 m}\Delta + V(\vec{x}) \right) \psi' \, , \tag{2.3}$$

for the choices

$$\gamma = -\frac{2mD}{\hbar}\left(1 - \frac{4m}{\hbar}D'c_2 - \frac{4m^2D^2}{\hbar^2} \right)^{-\frac{1}{2}} , \quad \Lambda = \left(1 - \frac{4m}{\hbar}D'c_2 - \frac{4m^2D^2}{\hbar^2} \right)^{-\frac{1}{2}} . \tag{2.4}$$

However, it should be noted that for non-integer values of Λ, the map N is not actually well-defined by Eq. (2.2) for all ψ. The statements in this paper therefore depend, in some cases, on an appropriate selection of wave functions.

The transformations N are *local*, in that they depend only on the values of ψ and $\bar{\psi}$. They also respect the *projective* structure of the Hilbert space $\mathcal{H} = L^2(I\!R^3, d^3x)$; i. e. for any complex number c,

$$N_{(\Lambda,\gamma)}(c\,\psi) = |c| e^{i(\gamma \ln |c| + \Lambda \arg c)} N_{(\Lambda,\gamma)}(\psi) \, , \tag{2.5}$$

whence $(c\,\psi)'$ belongs to the same ray as ψ'. Furthermore the transformations leave the symplectic structure $\omega = \delta\psi \wedge \delta\bar{\psi}$ on the Hilbert space $\mathcal{H}$ invariant up to a factor:

$$\begin{aligned} N^*_{(\Lambda,\gamma)}\omega \; &= \left(\frac{\partial N_{(\Lambda,\gamma)}}{\partial \psi}\delta\psi + \frac{\partial N_{(\Lambda,\gamma)}}{\partial \bar{\psi}}\delta\bar{\psi} \right) \wedge \overline{\left(\frac{\partial N_{(\Lambda,\gamma)}}{\partial \psi}\delta\psi + \frac{\partial N_{(\Lambda,\gamma)}}{\partial \bar{\psi}}\delta\bar{\psi} \right)} \\ &= \left(\frac{\partial N_{(\Lambda,\gamma)}}{\partial \psi}\frac{\partial \bar{N}_{(\Lambda,\gamma)}}{\partial \bar{\psi}} - \frac{\partial N_{(\Lambda,\gamma)}}{\partial \bar{\psi}}\frac{\partial \bar{N}_{(\Lambda,\gamma)}}{\partial \psi} \right) \delta\psi \wedge \delta\bar{\psi} \\ &= \Lambda\omega \end{aligned} \tag{2.6}$$

This gives a Hamiltonian formulation for the Ehrenfest family $\mathcal{F}_D^{Ehr}$, as has been noted elsewhere[4, 5], and establishes a connection to the framework of Weinberg[9].

The set $\mathcal{N} := \{N_{(\Lambda,\gamma)}\}$ of these nonlinear transformations obeys the group law of the affine group *Aff*(1) in one dimension,

$$N_{(\Lambda_1,\gamma_1)} \circ N_{(\Lambda_2,\gamma_2)} = N_{(\Lambda_1\Lambda_2,\,\Lambda_1\gamma_2+\gamma_1)} \, . \tag{2.7}$$

A further, essential property of $N \in \mathcal{N}$ is, of course, that it leaves the probability density invariant:

$$\rho'(\vec{x},t) = \rho(\vec{x},t)\,, \tag{2.8}$$

where ρ' is the density transformed under N. In a certain sense N is a nonlinear generalization of a linear, $U(1)$-gauge transformation. We shall call $\mathcal{N}$ the set of *local projective nonlinear gauge transformations*.

3. GAUGE INVARIANCE AND REPARAMETERIZATION

The transformation of the current $\vec{J}'$ under $N \in \mathcal{N}$ is given by

$$\vec{J}' := \mathrm{Im}\left(\bar{\psi}'\vec{\nabla}\psi'\right) = \rho'\vec{\nabla}\arg\psi' = \Lambda\vec{J} + \frac{\gamma}{2}\vec{\nabla}\rho\,. \tag{3.1}$$

In order to show the invariance of the family $\mathcal{F}_D$ of equations (1.2), we rewrite $\mathcal{F}_D$ wholly in terms of densities and currents. Using the expansion of the Laplacian $\Delta\psi = \{iR_1[\psi] + (1/2)R_2[\psi] - R_3[\psi] - (1/4)R_5[\psi]\}\,\psi$, we obtain the general form,

$$i\partial_t\psi = i\sum_{j=1}^{2}\nu_j R_j[\psi]\psi + \sum_{j=1}^{5}\mu_j R_j[\psi]\psi + \mu_0 V\psi. \tag{3.2}$$

From (2.2), (2.8) and (3.1), we deduce that $\psi' = N_{(\Lambda,\gamma)}(\psi)$ again fulfills (3.2), but with primed parameters:

$$\begin{gathered}
\nu_1' = \frac{\nu_1}{\Lambda}\,, \quad \nu_2' = -\frac{\gamma}{2\Lambda}\nu_1 + \nu_2\,, \\
\mu_1' = -\frac{\gamma}{\Lambda}\nu_1 + \mu_1\,, \quad \mu_2' = \frac{\gamma^2}{2\Lambda}\nu_1 - \gamma\nu_2 - \frac{\gamma}{2}\mu_1 + \Lambda\mu_2\,, \quad \mu_3' = \frac{\mu_3}{\Lambda} \\
\mu_4' = -\frac{\gamma}{\Lambda}\mu_3 + \mu_4\,, \quad \mu_5' = \frac{\gamma^2}{4\Lambda}\mu_3 - \frac{\gamma}{2}\mu_4 + \Lambda\mu_5\,, \quad \mu_0' = \Lambda\mu_0\,.
\end{gathered} \tag{3.3}$$

Thus we have that the 8-parameter family $\mathcal{F}$ of Eq. (3.2) is invariant under the action of $\mathcal{N}$; i.e., under the action of the affine group *Aff*(1). An appropriate description of $\mathcal{F}$ is by means of the *orbits* of *Aff*(1); since there are 2 group parameters, we look next for 6 functionally independent parameters that are invariant under *Aff*(1). These are the gauge invariants. After some calculations, we obtain gauge-invariant parameters:

$$\begin{gathered}
\iota_1 = \nu_1\mu_2 - \nu_2\mu_1\,, \quad \iota_2 = \mu_1 - 2\nu_2\,, \quad \iota_3 = 1 + \mu_3/\nu_1\,, \quad \iota_4 = \mu_4 - \mu_1\mu_3/\nu_1\,, \\
\iota_5 = \nu_1(\mu_2 + 2\mu_5) - \nu_2(\mu_1 + 2\mu_4) + 2\nu_2^2\mu_3/\nu_1\,, \quad \iota_0 = \nu_1\mu_0\,.
\end{gathered} \tag{3.4}$$

Now if we like we can choose ν_1 and μ_1 as our group parameters ($\nu_1 \neq 0$); this condition is fulfilled for all modifications of the linear Schrödinger equation, as ν_1 derives from the Laplacian in the Schrödinger equation. Inverting (3.4), we have

$$\begin{gathered}
\nu_2 = \frac{1}{2}(\mu_1 - \iota_2)\,, \quad \mu_2 = \frac{1}{2}\nu_1^{-1}(2\iota_1 - \iota_2\mu_1 + \mu_1^2)\,, \\
\mu_3 = (\iota_3 - 1)\nu_1\,, \quad \mu_4 = \iota_4 - \mu_1 + \iota_3\mu_1 \\
\mu_5 = \frac{1}{2}\nu_1^{-1}\left(\iota_5 - \iota_1 + \iota_4(\mu_1 - \iota_2) + \frac{1}{2}(\mu_1^2 - \iota_2^2)(\iota_3 - 1)\right)\,, \quad \mu_0 = \nu_1^{-1}\iota_0\,.
\end{gathered} \tag{3.5}$$

With this reparameterization the family of nonlinear Schrödinger equations is foliated in leaves characterized by $\iota_0, \ldots, \iota_5$, of subfamilies depending on the two group parameters

ν_1, μ_1. The group of nonlinear gauge transformations $\mathcal{N}$ acts effectively on each leaf of the foliation. Because of (2.8), the time-evolving probability density for all points (ν_1, μ_1) in a given leaf is the same.

Let us identify the gauge invariants for some special leaves:

a. The linear Schrödinger equation corresponds to the values $\nu_1 = -\hbar/2m$, $\mu_2 = -\hbar/4m$, $\mu_3 = \hbar/2m$, $\mu_5 = \hbar/8m$, $\mu_0 = 1/\hbar$, and $\nu_2 = \mu_1 = \mu_4 = 0$. We then have only two nonvanishing gauge invariants:

$$\iota_1 = \frac{\hbar^2}{8m^2}\,, \quad \iota_0 = -\frac{1}{2m} \;\;(\text{ for } V \not\equiv 0)\,. \tag{3.6}$$

Note that μ_0 and ι_0 are indeterminate if $V \equiv 0$. So $\hbar$ and m (or their quotient, in the free case) are gauge-invariant quantities for the family of equations.

b. The Ehrenfest subfamily $\mathcal{F}_D^{Ehr}$ corresponds to the values $\nu_1 = -\hbar/2m$, $\nu_2 = D/2$, $\mu_1 = D$, $\mu_2 = -\hbar/4m + c_2 D'$, $\mu_3 = \hbar/2m$, $\mu_4 = -D$, $\mu_5 = \hbar/8m - c_2 D'/2$, and $\mu_0 = 1/\hbar$. Again there are just two nonzero gauge invariants: ι_0 as before, and

$$\iota_1 = \frac{\hbar^2}{8m^2} - c_2\frac{\hbar D'}{2m} - \frac{D^2}{2}\,. \tag{3.7}$$

Here we rediscover the linearization of the Ehrenfest family, as when the right-hand side of (3.7) is positive, we can introduce a new constant $\hbar'$ such that a linear Schrödinger equation with $\hbar'$ replacing $\hbar$ is contained in the orbit.

c. For the more general, Galilei (Schrödinger) invariant subfamily[4, 10] $\mathcal{F}_D^{Gal}$ of $\mathcal{F}_D$, characterized by the conditions $c_1 + c_4 = c_3 = 0$, we have the values: $\nu_1 = -\hbar/2m$, $\nu_2 = D/2$, $\mu_1 = c_1 D'$, $\mu_2 = -\hbar/4m + c_2 D'$, $\mu_3 = \hbar/2m$, $\mu_4 = -c_1 D'$, $\mu_5 = \hbar/8m + c_5 D'$, and $\mu_0 = 1/\hbar$. Now we get four nonvanishing gauge invariants, taking independent values: ι_0, ι_1, ι_2, and ι_5.

4. FINAL REMARK

We have noted that the transformations linearizing the Ehrenfest subfamily $\mathcal{F}_D^{Ehr}$ of a family $\mathcal{F}_D$ of nonlinear Schrödinger equations can be viewed as generalizing the usual $U(1)$-gauge transformations, and act as a gauge group $\mathit{Aff}(1)$ on the parameter space of the family. We have calculated a parameterization by means of the gauge invariants, together with the group parameters of $\mathit{Aff}(1)$.

In connection with a more physical interpretation of our foliation of $\mathcal{F}_D$, we quote a remark by Feynman and Hibbs[11]:

> *Indeed all measurements of quantum-mechanical systems could be made to reduce eventually to position and time measurements. Because of this possibility a theory formulated in terms of position measurements is complete enough in principle to describe all phenomena.* (p. 96)

If one adopts this point of view, quantum theories for which the wave functions give the same probability density in space and time are "in principle" equivalent. Hence the leaves of our foliation consist of sets of "in principle" equivalent quantum-mechanical evolution equations.

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